Singular Laplacian growth

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The general equations of motion for two-dimensional singular Laplacian growth are derived using the conformal mapping method. In the singular case, all singularities of the conformal map are on the unit circle and the map is a degenerate Schwarz-Christoffel map. The equations of motion describe the motions of these singularities. Despite the typical fractal-like outcomes of Laplacian growth processes, the equations of motion are shown to be *not* particularly sensitive to initial conditions. It is argued that the sensitivity of this system derives from a different cause, the nonuniqueness of solutions to the differential system. By a mechanism of singularity creation, every solution can become more complex, even in the absence of noise, without violating the growth law. These processes are permitted, but are not required, meaning that the equation of motion does not determine the motion, even locally. [S1063-651X(98)09003-5]

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I. INTRODUCTION

Laplacian growth, growth of a region D along the gradient of its external Green's function, is a model for a number of growth processes that occur in nature, among them growth by electrodeposition [1], diffusion-limited aggregation [2], and viscous fingering at fluid-fluid interfaces [3]. These natural processes exhibit very complicated morphologies, as does the mathematical model. In spite of the large amount of work that has been done, one still has the feeling that there is something mysterious about Laplacian growth. In particular, its extreme sensitivity to perturbations makes it difficult to interpret experiments, real or numerical.

A special role is played in the two-dimensional problem by the conformal mapping method, as developed in [4–8], for example. This method gives insight difficult to attain otherwise. It is especially simple in radial geometry. In outline, one parametrizes the growing two-dimensional region D, thought of as occupying a bounded, simply connected region in the complex w plane, by the conformal map

$$w = G(z) \tag{1}$$

which takes the exterior of the unit disk $|z|>1$ onto the exterior of D. The growth of D is then represented by the time dependence of the conformal map G. Since G is holomorphic in $|z|>1$ and its dependence at infinity is also prescribed, G may in turn be parametrized by its singularities in the unit disk $|z|\leqslant 1$. The growth becomes the dynamics of those singularities. This method eliminates sources of inaccuracy that are unavoidable in other methods, for example, the statistical noise that accompanies diffusion-limited aggregation simulations or some of the roundoff and truncation errors of more straightforward integration methods. That does not make the conformal mapping method necessarily more realistic, of course. Indeed, the noise in other numerical methods may model actual physical noise and hence be desirable if one's aim is to model particular examples of Laplacian-like growth. With the conformal mapping method we aim rather to strip the problem down to its simplest form to see what remains and is common to all such processes.

In Ref. [7] this process of stripping down was taken one level further and an unexpected phenomenon came to light. This case might be called singular Laplacian growth because it is the limiting case in which all the singularities of G are *on* the unit circle $|z|=1$. In this case G degenerates to a Schwarz-Cristoffel map onto the exterior of a degenerate polygon D of zero area (i.e., D looks like a tree graph). In this limit the dynamics becomes one dimensional and can be understood completely. The surprise is that the dynamics allows the singularities of G to split and proliferate, but it does not require this. That is, while the dynamics is formally given by differential equations, the solutions, for given initial data, are not unique. The comment was made in Ref. [7] that this appears ''unlike any other physical model.'' In particular, it is not the same as being very sensitive to initial conditions, as one might have assumed. In fact, as we show below, singular Laplacian growth is not at all sensitive in this way. Its peculiarities have a different origin.

Reference [7] gave only the simplest example (in which all maps could be written down explicitly) and not a general computable theory. The present paper gives the general theory.

II. DYNAMICS OF SINGULARITIES

Let $w = G(t,z)$ be a time-dependent conformal map of the exterior of the unit disk in the z plane onto the exterior of the domain D in the w plane. G has the form

$$G(t,z) = z\sum_{k=0}^{\infty} c_k(t) z^{-k}. \tag{2}$$

Suppose for the moment that ∂D, the boundary of D, is an analytic Jordan curve so that there is no difficulty in defining

$$g(t,\theta) = G(t,e^{i\theta}). \tag{3}$$

As shown by Shraiman and Bensimon [5], Laplacian growth implies that the conformal map has time dependence given by

$$\frac{\partial g}{\partial t} = -i\frac{\partial g}{\partial \theta} L\left(\left|\frac{\partial g}{\partial \theta}\right|^{-2}\right), \tag{4}$$

where, if

$$\left|\frac{\partial g}{\partial \theta}\right|^{-2} = \sum_{k=-\infty}^{\infty} d_k e^{-ik\theta}, \tag{5}$$

then

$$L\left(\left|\frac{\partial g}{\partial \theta}\right|^{-2}\right) = d_0 + 2\sum_{k=1}^{\infty} d_k e^{-ik\theta}. \tag{6}$$

Define new scaled variables

$$a_k = c_k/c_0, \quad b_k = d_k/d_0 \tag{7}$$

and a new independent variable $s(t)$ by

$$ds/dt = d_0. \tag{8}$$

In terms of these variables, Eq. (4) becomes

$$\frac{da_k}{ds} = (k-2)a_k + 2\sum_{j=0}^{k} (1-j)a_j b_{k-j}. \tag{9}$$

Because of the rescaling in Eqs. (7) and (8), Eq. (9) continues to make sense even in the limit as singularities of the conformal map move onto the unit circle. For example, $|b_k| \leqslant 1$ for all k, even though d_k blows up. We define the b_k's, in case there are singularities on the unit circle, to have their limiting values as the singularities move onto the unit circle from the inside. With this understanding, Eq. (9) describes Laplacian growth, both singular and nonsingular, in terms of the scaled mapping function

$$H(s,z) = \frac{G(t(s),z)}{c_0(t(s))} = z\sum_{k=0}^{\infty} a_k(s) z^{-k}. \tag{10}$$

In the singular case, in which all singularities of H are on the unit circle, H is a Schwarz-Christoffel map onto a degenerate polygon and therefore its derivative has the form

$$\frac{\partial H}{\partial z} = \prod_{j=1}^{M} (1-e^{i\beta_j}/z)^{\alpha_j} \prod_{k=1}^{N} (1-e^{i\gamma_k}/z). \tag{11}$$

As a conformal map, H has singularities at points on the unit circle that we have called β_j and γ_k. The image of an arc of the unit circle under H, so long as it does not contain a singularity, is a straight line segment. At the singularity β_j, however, the image line turns through the angle $\alpha_j\pi$, which may be either positive (counterclockwise) or negative (clockwise). It is understood that $|\alpha_j| < 1$. At the singularity γ_k the image turns through the angle π, i.e., the line retraces itself; see Fig. 1. The γ singularity may seem to be only a special case of the β singularity, corresponding to $\alpha = 1$, but we have distinguished it because growth takes place entirely at the γ singularities: the γ's are different. (We use the following notation: α is an angle, β a branch point, and γ a growth tip. The α_j of this paper was called $\alpha_j - 1$ in [7].)

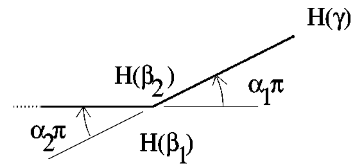

FIG. 1. The image under a degenerate Schwarz-Christoffel map of an arc containing singularities $\ldots, \beta_1, \gamma, \beta_2, \ldots$. The image comes in from the left, turns through an angle $\alpha_1\pi$ at $H(\beta_1)$, stops and reverses direction at $H(\gamma)$, turns through an angle $\alpha_2\pi$ at $H(\beta_2)$, and exits on the left. In this example, a kink is shown in a growing needle. Here $\alpha_1 = -\alpha_2$.

The α_j, β_j, and γ_k are by no means arbitrary. First, because the image polygon turns through a total angle 2π, one must have, since there are M branch points and N growth tips,

$$N + \sum_{j=1}^{M} \alpha_j = 2. \tag{12}$$

Second, because the image polygon looks like a tree graph, each edge of the image is traversed twice, once in each direction. This means that the integrals of $\partial H/\partial z$ along arcs of the unit circle from one singularity to the next, which are singular integrals, must cancel in pairs, an intricate condition on the positions of the β's and γ's.

Suppose at time $s = 0$ we have such a conformal map H. The equation of motion for H [Eq. (9)] should be recast as equations of motion for the singularities of H. The a_k's of Eq. (9) are just the coefficients in the power series for H, according to Eq. (10), but we still need the b_k's. To compute the b_k's, using Eqs. (5) and (7), we must Fourier transform $|\partial H/\partial z|^{-2}$, restricted to the unit circle, with the singularities displaced slightly inside. The Fourier transform integrals are dominated by the γ singularities, and as the singularities move onto the unit circle, the entire contribution comes from them. Thus there is a simple formula for b_k,

$$b_k = \sum_{j=1}^{N} v_j e^{ik\gamma_j}. \tag{13}$$

Here the weights v_j are determined by

$$v_j \propto \lim_{z \to e^{i\gamma_j}} \left|\frac{\partial H/\partial z}{z - e^{i\gamma_j}}\right|^{-2}, \tag{14}$$

with the constant of proportionality determined by the normalization

$$\sum_{j=1}^{N} v_j = b_0 = 1. \tag{15}$$

Define the function

$$B(z) = \sum_{k=0}^{\infty} b_k z^{-k} = \sum_{j=1}^{N} \frac{v_k}{1 - e^{i\gamma_j}/z}. \tag{16}$$

Then multiplying each side of Eq. (9) by z^{-k+1} and summing over k gives

$$\frac{\partial H}{\partial s} = -H + (2B-1)z\frac{\partial H}{\partial z}. \tag{17}$$

It is $\partial H/\partial z$ rather than H that we know explicitly, so take the derivative of Eq. (17) with respect to z. The left-hand side becomes

$$\frac{\partial^2 H}{\partial s \partial z} = -i\frac{\partial H}{\partial z}\left(\sum_{j=1}^{M}\frac{\alpha_j e^{i\beta_j}/z}{1-e^{i\beta_j}/z}\frac{d\beta_j}{ds} + \sum_{k=1}^{N}\frac{e^{i\gamma_k}/z}{1-e^{i\gamma_k}/z}\frac{d\gamma_k}{ds}\right) \tag{18}$$

and the right-hand side becomes an explicitly known expression. One can now cancel the factor $-i\partial H/\partial z$ in all terms. Multiplying by $1-e^{i\beta_j}/z$ and taking the limit as $z \to e^{i\beta_j}$ isolates $d\beta_j/ds$, and similarly multiplying by $1-e^{i\gamma_k}/z$ and taking the limit as $z \to e^{i\gamma_k}$ isolates $d\gamma_k/ds$. The result, after algebraic simplification, using Eqs. (15) and (12), is

$$\frac{d\beta_j}{ds} = \sum_{k=1}^{N} v_k \cot\left(\frac{\beta_j - \gamma_k}{2}\right), \tag{19}$$

$$\frac{d\gamma_k}{ds} = v_k \sum_{j=1}^{M} \alpha_j \cot\left(\frac{\gamma_k - \beta_j}{2}\right) + \sum_{j \neq k} (v_k + v_j)\cot\left(\frac{\gamma_k - \gamma_j}{2}\right). \tag{20}$$

We can also note that

$$v_k = w_k/W, \tag{21}$$

where

$$w_k = \prod_{j=1}^{M} \sin^{-2\alpha_j}\left(\frac{\gamma_k - \beta_j}{2}\right)\prod_{j \neq k} \sin^{-2}\left(\frac{\gamma_k - \gamma_j}{2}\right), \tag{22}$$

$$W = \sum_{k=1}^{N} w_k. \tag{23}$$

It is understood in Eq. (22) that w_k is real and positive. Equations (19)–(23) represent the dynamics of singular Laplacian growth as an autonomous system of ordinary differential equations (ODE's).

Remarkably, this system is a kind of gradient system:

$$\frac{d\beta_j}{ds} = -\frac{1}{\alpha_j}\frac{\partial \ln W}{\partial \beta_j}, \tag{24}$$

$$\frac{d\gamma_k}{ds} = -\frac{\partial \ln W}{\partial \gamma_k}. \tag{25}$$

This is gradient flow in the space of parameters $(\beta_1, \ldots, \beta_M, \gamma_1, \ldots, \gamma_N)$ endowed with the metric tensor

$$g = \mathrm{diag}(\alpha_1, \ldots, \alpha_M, 1, \ldots, 1). \tag{26}$$

Since, according to Eq. (12), the α's are negative, on average, if $N > 2$, this metric is indefinite. The gradient flow is toward certain critical points of $\ln W$ that are not minima. These critical points are the equilibria of singular Laplacian growth. They can be found by integrating the system of equations (19)–(23). Even if the starting state does not satisfy all the conditions described after Eq. (12), it will still approach a state that does satisfy them. We describe these equilibrium states more precisely below. This stability of the flow, which is a familiar property of gradient flows, is an indication that singular Laplacian growth is *not* sensitive to initial conditions, contrary to what one might have expected, and hence that the peculiar sensitivity of Laplacian growth in general either has been lost in the passage to the singular case or arises from some other cause. We suggest below that this other cause is the nonuniqueness property of the system, still to be described.

The dynamics of singularities described by Eqs. (19)–(23) can be pictured very simply. The β's are repelled by the γ's on the unit circle and the γ's repel each other. The strength with which each γ_k repels other singularities is given by the corresponding v_k (always greater than 0). The β's, on the other hand, do not interact directly with each other. No singularity can pass through another one; they always keep the same order around the unit circle. Those β's between any two adjacent γ's are, however, driven by them toward some intermediate point where, in effect, they coalesce into a single effective β, characterized by a single effective α, which is the sum of all the contributing α's. The way a β singularity approaches its limit position is the way $x(s)$ approaches 0 in

$$dx/ds = -x^2, \tag{27}$$

namely,

$$x(s) = x_0/(1 + x_0 s), \tag{28}$$

that is, it takes an infinite time. By the second derivative test, there is only one equilibrium position for the β's between each pair of γ's. Here all the β's will collect. Thus, in the limit as $s \to \infty$ the equilibrium states of singular Laplacian growth have γ's and β's alternating and look like N needles radiating from a single central point. One can even write a formula in closed form for H in this case,

$$H = z\prod_{k=1}^{N} (1 - e^{i\beta_k}/z)^{\alpha_k + 1}, \tag{29}$$

where the β's and α's are the effective ones. We can also understand this outcome, in a more physical way, by realizing that the continual rescaling means all internal structure shrinks to a point, leaving only the growth tips as visible features. What is not obvious from this description, but is observed, is that typically some of the γ's are entrained with the β's and coalesce with them (where they contribute $+1$ to the effective α). This amounts to the scaling away of needles. It turns out that the generic stable equilibria have $N \leq 3$. If the initial configuration has $N > 3$ and is the least bit asymmetrical, some of the growing tips lose out in the competition to grow and disappear as $s \to \infty$, leaving only three (or fewer) needles in the limit.

This result might appear puzzling since it seems to imply that Laplacian growth should be a process of *simplification*, contrary to the increasing complexity that is observed, and is the whole motivation for studying it. That puzzle is resolved in the next section.

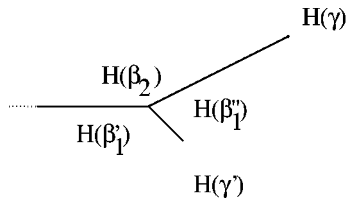

FIG. 2. A new needle may grow on the outside of the corner in Fig. 1. Here the singularity β_1 has split into two branch points β_1' and β_1'' and a new growth tip γ'.

III. NONUNIQUENESS OF THE DYNAMICS

If one reverses the sense of time and integrates the system backward, the repelling character of the γ's becomes an attraction. In particular, two β's adjacent on either side of a γ may be attracted to it, move toward it, and coalesce with it, essentially annihilating, leaving just the γ. Unlike the coalescence described at the end of the preceding section, which takes an infinite time, this coalescence occurs *in a finite time*. Roughly, one can estimate from Eq. (19) that a β approaches a nearby γ the way $x(s)$ approaches 0 in

$$dx/ds = 1/x \tag{30}$$

(integrating backward from $s=0$), namely,

$$x(s) = \sqrt{x_0^2 + 2s}. \tag{31}$$

At times earlier than the coalescence time $-x_0^2/2$, the β singularities were simply not present. If one now examines this solution to the system with the usual forward sense of time, one sees, at some arbitrary time $-x_0^2/2$, two β singularities suddenly produced on either side of a γ, which had not been there before. To satisfy Eq. (12), the α's that characterize these β's must add to zero. The geometrical effect of this process is that a kink of deviation angle α suddenly appears in the growing needle represented by γ, like the kink shown in Fig. 1, which might have formed from a single straight needle. This kinking may happen at any arbitrary time. A more careful argument (in the Appendix) says that if a kink forms at γ at $s=0$, the leading behavior in the motion of singularities is

$$\gamma - \beta_1 \propto \sqrt{\frac{1+\alpha_1}{1+\alpha_2}} s^{1/2}, \tag{32}$$

$$\gamma - \beta_2 \propto -\sqrt{\frac{1+\alpha_2}{1+\alpha_1}} s^{1/2}, \tag{33}$$

with $\alpha_1 = -\alpha_2$.

In addition, a second kind of coalescence is seen in backward integration, in which two β's, with angles α_1 and α_2 on either side of a γ, coalesce to leave a single β with angle $1+\alpha_1+\alpha_2$. This happens only if $1+\alpha_1+\alpha_2>0$ and $\alpha_1+\alpha_2<0$. Geometrically it corresponds to the shrinking away of a needle in a finite time (the growth tip γ is lost), on the outside of a corner of angle $1+\alpha_1+\alpha_2$. What it means in forward integration is that at any time a needle may begin *growing* on the outside of a corner, as in the process that takes Fig. 1 to Fig. 2. The motion of singularities in this case, in leading order, is

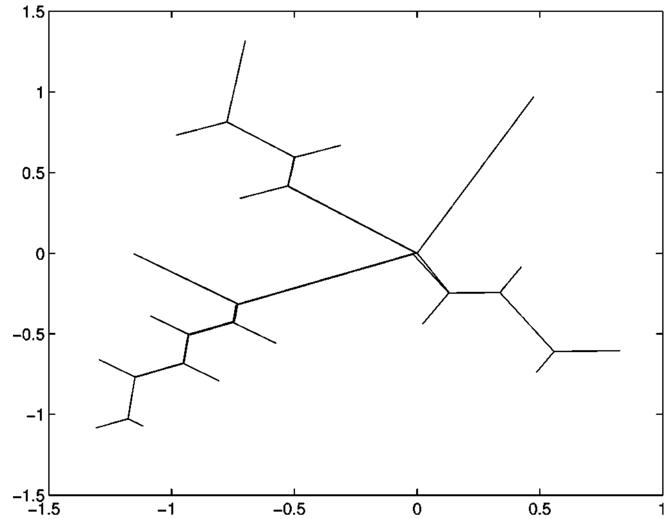

FIG. 3. At each time interval $\Delta s = 0.1$ the growing tip with the largest strength is split. The initial configuration was four random needles radiating from a point, but the growth law is completely deterministic. The failure of the image to retrace itself precisely is a measure of the numerical error in the method after about 40 branch points have been generated and moved according to the law of singular Laplacian growth.

$$\gamma - \beta_1 \propto \sqrt{\frac{1+\alpha_1}{1+\alpha_2}} s^{1/2(1+\alpha_1+\alpha_2)}, \tag{34}$$

$$\gamma - \beta_2 \propto -\sqrt{\frac{1+\alpha_2}{1+\alpha_1}} s^{1/2(1+\alpha_1+\alpha_2)}, \tag{35}$$

with $1+\alpha_1+\alpha_2>0$, $\alpha_1+\alpha_2<0$. (In Ref. [7] the factor $1+\alpha_1+\alpha_2$ in the exponent's denominator mistakenly appeared in the numerator.)

IV. DISCUSSION

The observations of Sec. III mean that the system of ODE's (19)–(23), although appearing unremarkable, has the peculiar property that its solutions are highly nonunique. New singularities can appear by the above two elementary processes at any time. In combination one has more complicated processes: A kink followed by a new needle at the outside of the new corner amounts to tip splitting, for example, and this can happen at any time. The equilibria described in Sec. II are never attained if such processes, which are allowed by the differential equation, continually intervene. Thus singular Laplacian growth supports complex nonequilibrium behavior after all.

It is interesting to see what the model looks like if one integrates it forward, introduces new singularities, integrates again, adds more singularities, etc. Examples are shown in Figs. 3 and 4, where symmetrical tip splitting was introduced at intervals of 0.1 time unit. To interpret the evolving positions of singularities in terms of the corresponding image region D, which is what is shown, it was necessary to integrate Eq. (11) numerically. Each edge is represented by a singular integral. These integrals were done by Gauss-Jacobi integration, as described by Trefethen in Ref. [9]. The accumulating error in these numerical integrals, as one steps

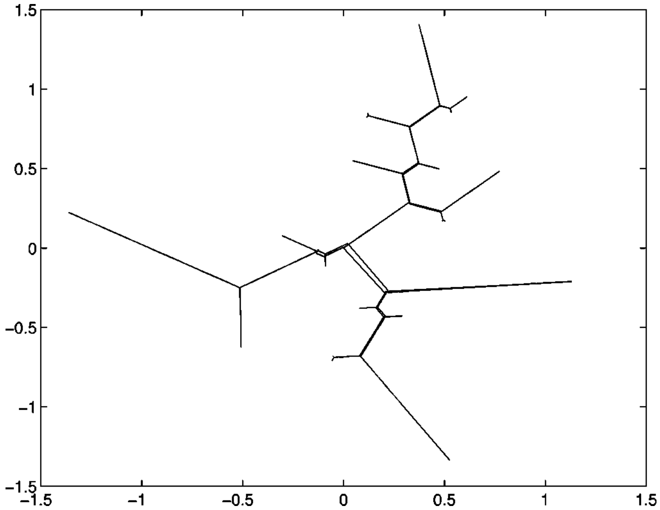

FIG. 4. At each time interval $\Delta s = 0.1$ a growing tip is randomly selected and split. The probability of a tip's being selected is proportional to its strength v. The initial configuration was three random needles radiating from a point. The failure of one of the edges to retrace itself accurately is the result of accumulated numerical error.

along each edge to the next, especially in light of the usual sensitivity of numerical conformal maps, might have been expected to produce nonsensical pictures, but in fact the numerical error (failure to retrace edges accurately) is just barely visible in these examples. (Eventually, of course, the accumulating error does become large, but the good numerical behavior of the system again makes the point that singular Laplacian growth is *not* particularly sensitive to error or noise. Its sensitivity to perturbations comes entirely through the nonuniqueness property.)

V. RELATION TO OTHER WORK

Most of those who have used the conformal mapping method have followed Shraiman and Bensimon [5] in restricting the derivative of the conformal map H to be a rational function. From some points of view this is a rather drastic restriction on the analytic structure of H. Whether it is a good enough representation of H to learn the full implications of the conformal mapping method is not clear. Arguments that the boundary value of H' can always be approximated by the boundary value of a rational function are not very convincing in a context where it is precisely the nature of the singularities that is the basis of the theory. It had already been noticed in Ref. [7] that branch points play an essential role in the singular theory. Nonetheless, an interesting comparison between the singular case and the rational case is possible.

An example is Ref. [8], in which Blumenfeld and Ball invent a mechanism of ''particle creation'' (i.e., singularity creation) to model tip splitting. In their model, since H' is rational, the only singularities are the zeros and poles of H'. The mechanism they propose is that a zero creates a second zero and a pole. The two zeros represent the two growing tips after the split and the pole represents the division between them.

Tip splitting in the singular theory, as described in Ref. [7] and in this paper, does not have to be invented (it is naturally and unavoidably part of the theory) and it looks slightly more complicated: A γ gives rise to three β's and another γ (as in Figs. 1 and 2). However, this amounts in fact to the same thing. The resulting two γ's are zeros of H' and, by the geometry of the situation, the three new α's add to -1. This means that the three β's, from a distance much greater than their mutual separation, look like a pole of H' [see Eq. (11)]. The mechanism proposed by Blumenfeld and Ball is thus a kind of smeared version of the singular mechanism, already described in Ref. [7]

It is especially remarkable that Blumenfeld and Ball invented their mechanism entirely on the basis of physical phenomenology and were unaware of Ref. [7]. Their mechanism of particle creation, although it is *ad hoc*, corresponds as precisely as it could have to the *only* mechanism in the singular theory for nontrivial dynamics. This suggests that the singular theory is close enough to real phenomenology to be useful and it does retain the essential features of Laplacian growth.

VI. GENERALITIES

To focus on the details of singular Laplacian growth is, to some extent, to sidestep a much bigger question: What is going on here with nonuniqueness? Are not differential equations supposed to have unique solutions? We all know textbook examples where uniqueness fails, but the failure occurs on some small set and for equations that would not arise in physics. Here are equations that arise in a system that has been much discussed in physics and uniqueness fails for every solution at every time. The least one can say is that the equations of motion do not determine the motion, even locally.

I believe this is actually mathematical *terra incognita*. Such equations do not even have a name. How would one characterize them generally? Are they in some sense common, or are they rare? I think of calling them ''fragile differential equations'' because, at least in this example, the nonuniqueness arises by the tendency of singularities to ''break apart,'' but perhaps a more general understanding would reveal that this name is somehow misleading. ''Fragile'' sounds a little bit like ''fractal,'' but is not the same, another reason I like the name.

On a more physical level, what does it mean for a physical system if it is described by equations that, in some limit, become ''fragile?'' A fragile system does not fully determine the evolution, but it does restrict it. What is the nature of the restriction? These seem like good questions for the future.

APPENDIX

We derive Eqs. (32)–(35), the leading behavior of singularities β_1, γ, β_2 when they are very close to each other (in that order) and not close to other singularities. Let α_1 and α_2 be the corresponding angle parameters and v the strength of γ. According to Eqs. (19) and (20), keeping only the most singular terms, in leading order they obey

$$\frac{d\beta_1}{ds} = \frac{2v}{\beta_1 - \gamma}, \tag{A1}$$

$$\frac{d\beta_2}{ds}=\frac{2v}{\beta_2-\gamma}, \qquad \text{(A2)}$$

$$\frac{d\gamma}{ds}=\frac{2v\alpha_1}{\gamma-\beta_1}+\frac{2v\alpha_2}{\gamma-\beta_2}. \qquad \text{(A3)}$$

Let

$$P=\gamma-\beta_1, \qquad \text{(A4)}$$

$$Q=\gamma-\beta_2. \qquad \text{(A5)}$$

Then, subtracting,

$$\frac{dP}{ds}=\frac{2v(1+\alpha_1)}{P}+\frac{2v\alpha_2}{Q}, \qquad \text{(A6)}$$

$$\frac{dQ}{ds}=\frac{2v\alpha_1}{P}+\frac{2v(1+\alpha_2)}{Q}. \qquad \text{(A7)}$$

Dividing, we have the homogeneous equation

$$\frac{dP}{dQ}=\frac{(1+\alpha_1)Q+\alpha_2 P}{\alpha_1 Q+(1+\alpha_2)P}, \qquad \text{(A8)}$$

which separates when written in terms of the variable P/Q. The complete solution, in implicit form, is

$$(P-Q)^{\nu_1}[(1+\alpha_2)P+(1+\alpha_1)Q]^{\nu_2}=\text{const}, \qquad \text{(A9)}$$

where

$$\nu_1=(1+\alpha_1+\alpha_2)/(2+\alpha_1+\alpha_2), \qquad \text{(A10)}$$

$$\nu_2=1/(2+\alpha_1+\alpha_2). \qquad \text{(A11)}$$

Since this is supposed to hold as P,Q approach zero, the only relevant value of the constant is zero. The solution $P=Q$ is not relevant to this situation since P and Q must have opposite sign. Thus

$$(1+\alpha_2)P+(1+\alpha_1)Q=0. \qquad \text{(A12)}$$

Using Eqs. (21)–(23) together with the fact, found in Eq. (A12), that $\gamma-\beta_1$ and $\gamma-\beta_2$ are simply proportional, we see that v is nonsingular if $\alpha_1+\alpha_2\geqslant 0$ and

$$v\sim P^{-2\alpha_1-2\alpha_2} \qquad \text{(A13)}$$

if $\alpha_1+\alpha_2<0$. Thus, from Eq. (A6),

$$dP/ds\sim P^{-1} \qquad \text{(A14)}$$

if $\alpha_1+\alpha_2\geqslant 0$, as in the rough argument of Sec. III, and

$$\frac{dP}{ds}\sim P^{-2\alpha_1-2\alpha_2-1} \qquad \text{(A15)}$$

if $\alpha_1+\alpha_2<0$. These results are summarized in Eqs. (32)–(35).

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